Magnetic flux Φ

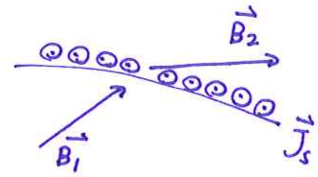
$$\Phi = \int_S \vec{B}\cdot d\vec{s} = \oint_C \vec{A}\cdot d\vec{l} \quad (\text{wb})$$

Magnetic Permeability

$$\vec{B} = \mu_0(\vec{H}+\vec{M}) = \mu_0(\vec{H}+\chi_m\vec{H}) = \mu_0\overbrace{(1+\chi_m)}^{\mu_r}\vec{H}$$

$$\vec{B} = \mu_0\mu_r\vec{H}$$

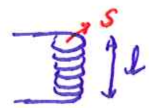
Magnetic Boundary Condition



$$\begin{cases} B_{1n} = B_{2n} \\ H_{2t} - H_{1t} = J_s \quad \left(\frac{A}{m}\right) \end{cases}$$

Self-inductance: $\quad L \equiv \dfrac{\Lambda}{I} \quad (H) \qquad \Lambda$: magnetic flux linkage $= N\int \vec{B}\cdot d\vec{s}$

For a solenoid: $\quad L = \mu \dfrac{N^2}{l} S$

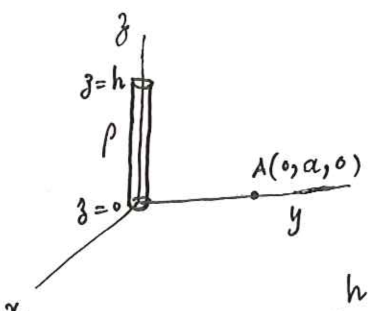


Mutual inductance:

$$L_{12} = \frac{\Lambda_{12}}{I_1} = \frac{N_2}{I_1}\int_{S_2}\vec{B_1}\cdot d\vec{s} \quad (H)$$

Magnetic Energy: $\quad W_m = \dfrac{1}{2}\displaystyle\int_V \mu H^2\, dv$

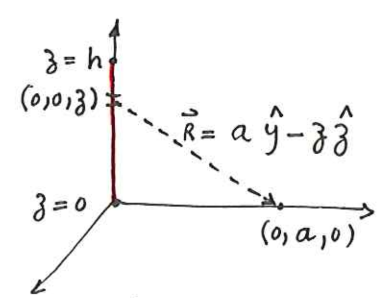
Example1: For a line of charge as depicted in the picture calculate the electric field at point A.



Solution: $\quad \vec{E} = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \frac{\rho\, dl}{R^3}\vec{R}$

$\vec{R} = a\hat{y} - z\hat{z}$

$R = (a^2+z^2)^{1/2}$



$$\rightarrow \vec{E} = \frac{\rho}{4\pi\varepsilon_0}\int_0^h \frac{dz}{(a^2+z^2)^{3/2}}(a\hat{y}-z\hat{z}) = \frac{\rho}{4\pi\varepsilon_0}\left[\hat{y}\int_0^h \frac{a\,dz}{(a^2+z^2)^{3/2}} - \hat{z}\int_0^h \frac{z\,dz}{(a^2+z^2)^{3/2}}\right]$$

$$= \frac{\rho}{4\pi\varepsilon_0}\left[\hat{y}\int_0^h \frac{a^2+z^2-z^2}{a(a^2+z^2)^{3/2}}dz - \hat{z}\frac{1}{2}\int_0^h \frac{2z\,dz}{(a^2+z^2)^{3/2}}\right]$$

$$= \frac{\rho}{4\pi\varepsilon_0}\left[\hat{y}\int_0^h \left(\frac{dz}{a(a^2+z^2)^{1/2}} - \frac{z^2\,dz}{a(a^2+z^2)^{3/2}}\right) + \hat{z}\,\frac{1}{(a^2+z^2)^{1/2}}\Bigg|_0^h\right]$$

$$= \frac{\rho}{4\pi\varepsilon_0}\left[\hat{y}\int_0^h d\left(\frac{z}{a(a^2+z^2)^{1/2}}\right) + \hat{z}\,\frac{1}{(a^2+z^2)^{1/2}}\Bigg|_0^h\right]$$

$$= \frac{\rho}{4\pi\varepsilon_0}\left[\hat{y}\,\frac{z}{a(a^2+z^2)^{1/2}} + \hat{z}\,\frac{1}{(a^2+z^2)^{1/2}}\right]\Bigg|_0^h$$

$$\vec{E} = \frac{\rho}{4\pi\varepsilon_0}\left[\hat{y}\,\frac{h}{a(a^2+h^2)^{1/2}} + \hat{z}\left(\frac{1}{(a^2+h^2)^{1/2}} - \frac{1}{a}\right)\right]$$

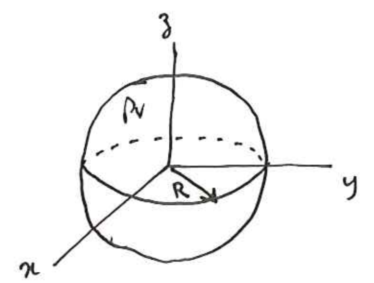
Example 2

Given $\vec{D} = \hat{x}\,2(x+y) + \hat{y}\,(3x-2y)$, find ρ_v.

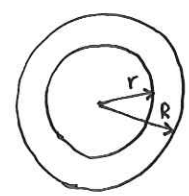
Solution: $\vec{\nabla}\cdot\vec{D} = \rho_v \rightarrow \rho_v = \frac{\partial}{\partial x}[2(x+y)] + \frac{\partial}{\partial y}(3x-2y) = 2-2 = 0$

Example 3

Find the electric field around a sphere of radius R with charge density ρ_v.



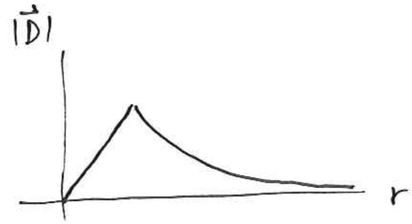
Solution: if $r<R$:



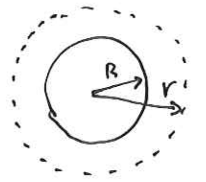
$$\int \vec{D}\cdot d\vec{s} = Q = \int \rho_v\,dv$$

$$D(4\pi r^2) = \rho_v\left(\tfrac{4}{3}\pi r^3\right)$$

$$\vec{D} = \rho_v\,\frac{r}{3}\,\hat{R}$$



if $r>R$:



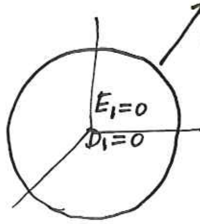
$$\int \vec{D}\cdot d\vec{s} = Q = \int \rho_v\,dv$$

$$D(4\pi r^2) = \rho_v\left(\tfrac{4}{3}\pi R^3\right)$$

$$\vec{D} = \rho_v\,\frac{R^3}{3r^2}\,\hat{R}$$

Example 4

A spherical metal ball of radius 2 cm is inside oil with $\varepsilon_r = 9$. If there exists an electric field of $\vec{E} = \hat{R}\, 3\cos\theta - \hat{\theta}\, 3\sin\theta$ $\left(\frac{V}{m}\right)$ in the oil, find the charge density on the ball.



$\vec{E}_2 = \hat{R}\, 3\cos\theta - \hat{\theta}\, 3\sin\theta$

$\varepsilon_2 = 9$

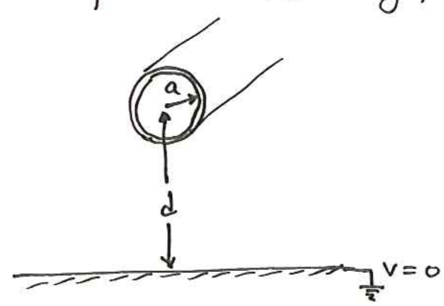
$\rightarrow \vec{D}_2 = (\hat{R}\, 27\cos\theta - \hat{\theta}\, 27\sin\theta)\, \varepsilon_0$

$$\nabla \cdot D = \rho \rightarrow \cancel{D_{1n}}^{\,0} - D_{2n} = \rho_s \rightarrow \rho_s = -D_{2n} = -\vec{D}_2 \cdot \hat{R} = -(\underbrace{\hat{R}\cdot\hat{R}}_{=1}\, 27\cos\theta - \underbrace{\hat{\theta}\cdot\hat{R}}_{=0}\, 27\sin\theta)\,\varepsilon_0$$

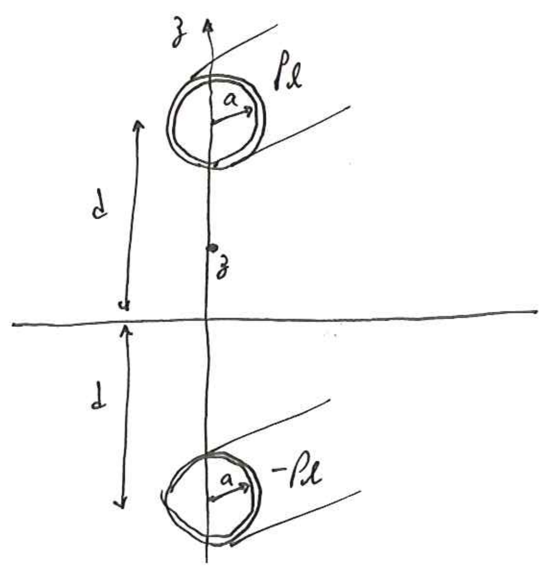
$$\rho_s = -27\cos\theta\, \varepsilon_0 \quad \left(\frac{C}{m^2}\right)$$

Example 5

Find the capacitance per unit length of an infinitely long conducting cylinder of radius a situated at a distance d from a conducting plate as shown here.

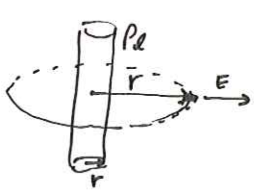


Solution: Using the image method, we have:



$$C = \frac{Q}{V} = \frac{Q}{-\int E \cdot dl}$$

For a cylinder we have:



$$\int D \cdot ds = Q = \int \rho_l \, dl$$

$$D\, 2\pi r l = \rho_l l \rightarrow \vec{D} = \frac{\rho_l}{2\pi r}\hat{r}$$

So for the top conductor, $\vec{E}$ at z is:

$\vec{D}_1 = -\hat{z}\dfrac{\rho_l}{2\pi(d-z)}$ and for the bottom conductor: $\vec{D}_2 = -\hat{z}\dfrac{\rho_l}{2\pi(d+z)}$. So the total $\vec{D}$ at z is:

$$\vec{D}(z) = \vec{D}_1 + \vec{D}_2 = -\hat{z}\frac{\rho_l}{2\pi}\left(\frac{1}{d-z} + \frac{1}{d+z}\right) \rightarrow \vec{E} = -\hat{z}\frac{\rho_l}{2\pi\varepsilon_0}\left(\frac{1}{d-z} + \frac{1}{d+z}\right)$$

So V between the two conductors is: $V = -\int_{z=-(d-a)}^{z=d-a} E \cdot dl$

$$\rightarrow V = \frac{\rho_l}{2\pi\varepsilon_0}\left(\int \frac{dz}{d-z} + \int \frac{dz}{d+z}\right) = \frac{\rho_l}{2\pi\varepsilon_0}\left(-\ln(d-z) + \ln(d+z)\right)\Big|_{-(d-a)}^{d-a}$$

$$= \frac{\rho_l}{2\pi\varepsilon_0}\ln\left(\frac{d+z}{d-z}\right)\Big|_{-(d-a)}^{d-a} = \frac{\rho_l}{2\pi\varepsilon_0}\left(\ln\frac{2d-a}{a} - \ln\frac{a}{2d-a}\right)$$

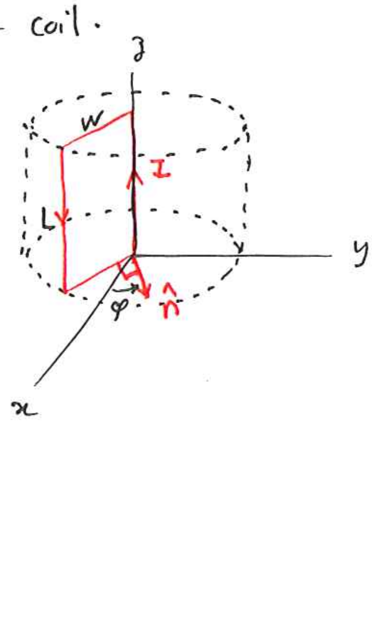
$$V = \frac{\rho_l}{\pi\varepsilon_0}\ln\frac{2d-a}{a}$$

Also: $Q = \rho_l L \rightarrow C = \frac{Q}{V} = \frac{\rho_l L}{\frac{\rho_l}{\pi\varepsilon_0}\ln\frac{2d-a}{a}} \rightarrow \frac{C}{L} = \frac{\pi\varepsilon_0}{\ln\frac{2d-a}{a}} \quad \left(\frac{F}{m}\right)$

<u>Example 6</u> A 20-turn rectangular coil of 30cm x 10cm is placed as shown in the picture in yz plane. If the coil which caries a current $I = 10A$ is in a magnetic flux density $\vec{B} = 2\times10^{-2}(\hat{x} + \hat{y}2)$ determine the torque on the coil.

Solution: we know $\vec{T} = \vec{m} \times \vec{B}$

$$\vec{m} = \hat{n}NIA = \hat{n}(20)(10)(0.3\times0.1) = \hat{n}6$$



$$\rightarrow \vec{T} = 6\hat{n}\times(\hat{x}+\hat{y}2)(2\times10^{-2})$$

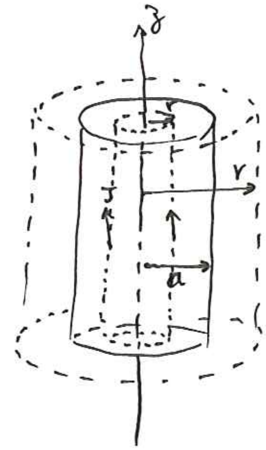
$$= 0.12(\hat{n}\times\hat{x} + \hat{n}\times\hat{y}2) = 0.12(-\hat{z}\sin\varphi + \hat{z}\cos\varphi 2)$$

$$\vec{T} = 0.12(2\cos\varphi - \sin\varphi)\hat{z}$$

* At what φ, the torque is maximum?

$$\frac{dT}{d\varphi} = 0 \rightarrow -2\sin\varphi - \cos\varphi = 0 \rightarrow \tan\varphi = -\frac{1}{2} \rightarrow \varphi = -26.57° \text{ or } 153.43°$$

Example 7 Find the magnetic field arround a cylindrical conductor carrying current density $\vec{J} = \hat{z} J_0 e^{-r}$ as shown for $r<a$.



$$\oint_c \vec{H} \cdot d\vec{l} = I = \int_s \vec{J} \cdot d\vec{s}$$

$$H(2\pi r) = \int_0^r J_0 e^{-r} 2\pi r \, dr = 2\pi J_0 \int_0^r r e^{-r} dr$$

$$= 2\pi J_0 \left(-re^{-r} + \int e^{-r} dr \right) \Big|_0^r$$

$$= 2\pi J_0 \left(-re^{-r} - e^{-r} \right) \Big|_0^r = 2\pi J_0 \left(-re^{-r} - e^{-r} + 1 \right)$$

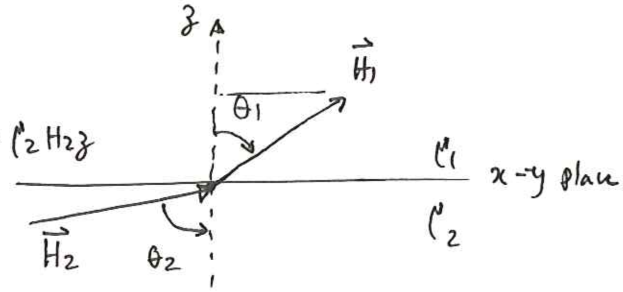
$$\Rightarrow \quad H = J_0 \left(\frac{1}{r} - e^{-r} - \frac{1}{r} e^{-r} \right) \qquad r \leq a$$

for $r>a$:

$$H(2\pi r) = \int_0^a J_0 e^{-r} 2\pi r \, dr = 2\pi J_0 \left(1 - ae^{-a} - e^{-a} \right)$$

$$\rightarrow \quad H = \frac{J_0}{r} \left(1 - ae^{-a} - e^{-a} \right) \qquad r \geq a$$

Example 8 In the picture $\vec{H}_1 = \hat{x} H_{1x} + \hat{y} H_{1y} + \hat{z} H_z$ and there is no surface current at the interface. Find $\vec{H}_2$, θ_1, θ_2.



$$\nabla \cdot B = 0 \rightarrow B_{1n} = B_{2n} \Rightarrow \mu_1 H_{1n} = \mu_2 H_{2n} \rightarrow \mu_1 H_{1z} = \mu_2 H_{2z}$$

$$\nabla \times H = 0 \rightarrow H_{1t} = H_{2t} \rightarrow \begin{cases} H_{1x} = H_{2x} \\ H_{1y} = H_{2y} \end{cases}$$

$$\rightarrow \quad \vec{H}_2 = \hat{x} H_{1x} + \hat{y} H_{1y} + \hat{z} \frac{\mu_1}{\mu_2} H_{1z}$$

$$\tan\theta_1 = \frac{H_{1t}}{H_{1n}} = \frac{\sqrt{H_{1x}^2 + H_{1y}^2}}{H_{1z}} \qquad \tan\theta_2 = \frac{\sqrt{H_{1x}^2 + H_{1y}^2}}{H_{1z}} \frac{\mu_2}{\mu_1} = \frac{\mu_2}{\mu_1} \tan\theta_1$$

$$\rightarrow \quad \frac{\tan\theta_1}{\mu_1} = \frac{\tan\theta_2}{\mu_2}$$